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A continued fraction solution to the problem of a single atom interacting with a single radiation mode in the electric dipole approximation

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Abstract. Difference equations are derived for the Laplace transforms of the time dependent probability amplitudes of a system consisting of a single photon interacting with a single two-level atom in the electric dipole approximation. The rotating wave approximation is not made. Formally exact continued fraction solutions of these difference equations are obtained and applied to the problems of stimulated and spontaneous emission and absorption of radiation.

1. Introduction

The simplest model for the interaction between radiation and matter is that in which a single two-level atom interacts with a single mode of the electromagnetic field. If the interaction between the atom and the field is assumed to be electric dipole, the Hamiltonian may be written in second quantized form as (using a system of units in which $\hbar = 1$)

$$H = (\sigma_3 + \tfrac{1}{2})\omega_0 + (a^\dagger a + \tfrac{1}{2})\omega + (\sigma_+ + \sigma_-)(ga^\dagger + g^* a). \tag{1}$$

If $|\alpha\rangle$ is the excited energy eigenstate of the isolated atom, and $|\beta\rangle$ is the ground state, then the 'spin' operators σ_3, σ_+ and σ_- are defined by the following relations:

$$\sigma_3|\alpha\rangle = \tfrac{1}{2}|\alpha\rangle, \qquad \sigma_3|\beta\rangle = -\tfrac{1}{2}|\beta\rangle \tag{2}$$

$$\sigma_+|\alpha\rangle = 0, \qquad \sigma_+|\beta\rangle = |\alpha\rangle \tag{3}$$

and

$$\sigma_-|\alpha\rangle = |\beta\rangle, \qquad \sigma_-|\beta\rangle = 0. \tag{4}$$

$a^\dagger$ is the creation operator for a photon of frequency ω, and ω_0 is the energy separation of the isolated atom's eigenstates. The coupling constant g is given by

$$g = -\mathrm{i}\left(\frac{\omega}{2\epsilon}\right)^{1/2} \boldsymbol{d} \cdot \boldsymbol{u} \tag{5}$$

where ϵ is the dielectric constant, $\boldsymbol{d}$ is a dipole matrix element, and $\boldsymbol{u}$ is the normal mode function for the field.

It is customary in calculations in quantum optics to make the rotating wave approximation (RWA) which involves neglecting the terms $ga^\dagger\sigma_+$ and $g^* a\sigma_-$ in (1). These

terms are the so called antiresonant terms. When this approximation is made, the resulting Hamiltonian can be solved exactly (Jaynes and Cummings 1963). Although exact solutions have been found to various multi-atom multi-mode generalizations of (1) in the RWA (Tavis and Cummings 1968, Mallory 1969, Walls and Barakat 1970, Scharf 1970, Walls 1971, Davidson and Kozak 1971, Swain 1972a, b, c, d), to the author's knowledge no exact solution of the Hamiltonian (1) has been published. In this paper we present exact continued fraction expressions for various quantities which determine the time-dependent behaviour of the system described by (1).

In § 2 we review the method of approach and in § 3 determine the fundamental difference equations. These are solved in § 4 for the specially simple case of spontaneous emission. The results of § 4 are then generalized in § 5 to obtain solutions for the problems of stimulated emission and absorption. In § 6 we give the general solutions of the fundamental difference equations.

2. The generalized Fourier transform approach

The method we adopt is that described by Pike and Swain (1971) which makes use of the generalized Laplace (or Fourier) transform approach (Titchmarsh 1937). However, the work of Jaynes and Cummings suggests that it is more profitable to work with probability amplitudes, rather than with probabilities directly, for this particular system. (If one writes down the equations of motion for the density matrix appropriate to (1) in the RWA, one obtains an infinite set of coupled equations, whereas the equations of motion for the probability amplitudes involve only two coupled equations, which are easily solved.) It is therefore necessary to adapt the approach of Pike and Swain slightly.

If $|a\rangle$ and $|b\rangle$ are two possible state vectors of the system at time $t = 0$, we shall be interested in calculating quantities of the form

$$\langle a|b(t)\rangle = \langle a|\, \mathrm{e}^{-\mathrm{i}Ht}|b\rangle, \tag{6}$$

which we shall refer to as probability amplitudes. They are clearly matrix elements of the time evolution operator, and a knowledge of all these amplitudes would furnish a complete description of the possible time evolutions of the system from any initial state. Furthermore, we have that $P_{a,b}(t)$, the probability that the system will be in state $|a\rangle$ at time t if it was in state $|b\rangle$ at time $t = 0$, is given by

$$P_{a,b}(t) = |\langle a|b(t)\rangle|^2. \tag{7}$$

We may write

$$\langle a|b(t)\rangle = \sum_{\xi} \langle a|\xi\rangle \exp(-\mathrm{i}E_{\xi}t)\langle\xi|b\rangle \tag{8}$$

$$= -\frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}E \sum_{\xi} \frac{\langle a|\xi\rangle\langle\xi|b\rangle}{E - E_{\xi}} \mathrm{e}^{-\mathrm{i}Et} \tag{9}$$

where $|\xi\rangle$ is an exact eigenstate of H, and E_{ξ} is the corresponding energy eigenvalue. In (9) the contour lies above the real axis and encloses all the poles of the integrand. If we define

$$L_E(\langle a|\,;|b\rangle) = \sum_{\xi} \frac{\langle a|\xi\rangle\langle\xi|b\rangle}{E - E_{\xi}} \tag{10}$$

it is easy to show that $L_E(\langle a|\,;|b\rangle)$ obeys the equation of motion

$$EL_E(\langle a|\,;|b\rangle) = \langle a|b\rangle + L_E(\langle a|H\,;|b\rangle). \tag{11}$$

Here $\langle a|H$ is the hermitean conjugate of the ket obtained by operating with H on $|a\rangle$. If the heirarchy of equations generated by (11) can be solved in some way, the probability coefficients may be obtained from the relation

$$\langle a|b(t)\rangle = -\frac{1}{2\pi i}\oint \mathrm{d}E L_E(\langle a|\,;|b\rangle)\,\mathrm{e}^{-iEt}. \tag{12}$$

3. Derivation of the difference equations

The evaluation of the quantity $\langle a|H$ which appears in the right hand side of the equation of motion (11) is facilitated if we write the Hamiltonian (1) in a different form. We do this by making use of the identities

$$|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| = 1 \tag{13}$$

and

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \tag{14}$$

where n is an eigenvalue of the number operator $a^\dagger a$. Utilizing (13) and (14) in (1) we find

$$H = \omega_0|\alpha\rangle\langle\alpha| + \sum_n n\omega|n\rangle\langle n| + \sum_n \sqrt{n+1}(|\alpha\rangle\langle\beta| + |\beta\rangle\langle\alpha|)(g|n+1\rangle\langle n| + g^*|n\rangle\langle n+1|). \tag{15}$$

(For future convenience we have omitted the zero point energy of the field.)

We wish to consider the time evolution from some initial state $|\psi\rangle$ which for the moment we leave unspecified. If we define the quantities

$$\alpha_n \equiv L_E(\langle\alpha|\langle n|\,;|\psi\rangle) \tag{16}$$

and

$$\beta_n \equiv L_E(\langle\beta|\langle n|\,;|\psi\rangle) \tag{17}$$

it is quite easy to show by using the Hamiltonian (15) with the equation of motion (11) that α_n and β_n satisfy the following recurrence relationships:

$$(E-\omega_0-n\omega)\alpha_n = a_n + g\sqrt{n}\beta_{n-1} + g^*\sqrt{n+1}\beta_{n+1} \tag{18}$$

$$(E-n\omega)\beta_n = b_n + g\sqrt{n}\alpha_{n-1} + g^*\sqrt{n+1}\alpha_{n+1} \tag{19}$$

where

$$\left.\begin{aligned} a_n &= (\langle\alpha|\langle n|)\,.\,|\psi\rangle \\ b_n &= (\langle\beta|\langle n|)\,.\,|\psi\rangle \end{aligned}\right\} \tag{20}$$

and $n = 0, 1, 2, 3, \ldots$. The terms a_n and b_n must satisfy the completeness relation

$$\sum_n (|a_n|^2 + |b_n|^2) = 1. \tag{21}$$

The set of equations (18), (19) and (20) provide a complete description of the time-dependent properties of the system given by (1) which is prepared in the arbitrary initial state $|\psi\rangle$, in terms of the initial matrix elements a_n and b_n. The α_n and β_n will in general be functions of all the a_n and b_n; consequently it is not surprising that these equations are cumbersome to solve in general.

If the rotating wave approximation is made, equations (18) and (19) reduce to

$$(E - \omega_0 - n\omega)\alpha_n^{(R)} = a_n + g^*\sqrt{n+1}\,\beta_{n+1}^{(R)} \tag{22}$$

$$(E - n\omega)\beta_n^{(R)} = b_n + g\sqrt{n}\,\alpha_{n-1}^{(R)} \tag{23}$$

which are easily solved without making any further approximations to give

$$\alpha_n^{(R)} = \frac{a_n\{E - (n+1)\omega\} + g^*\sqrt{n+1}\,b_{n+1}}{(E - \omega_0 - n\omega)\{E - (n+1)\omega\} - |g|^2(n+1)} \tag{24}$$

$$\beta_n^{(R)} = \frac{b_n\{E - \omega_0 - (n-1)\omega\} + g\sqrt{n}\,a_{n-1}}{\{E - \omega_0 - (n-1)\omega\}(E - n\omega) - |g|^2 n}. \tag{25}$$

Actually, the system (18) and (19) comprises two independent sets of equations, for if one takes $n = 0$ in (18) one finds that the quantities

$$\alpha_0, \alpha_2, \alpha_4, \ldots \quad \text{and} \quad \beta_1, \beta_3, \beta_5, \ldots \tag{26}$$

are coupled, whereas if one initially takes the value $n = 1$ in (18), one finds that the quantities

$$\alpha_1, \alpha_3, \alpha_5, \ldots \quad \text{and} \quad \beta_0, \beta_2, \beta_4, \ldots \tag{27}$$

are connected. This corresponds to the property that if, say, the system was initially in a state in which the atom was in its excited state with an odd number of photons present, then there are no transitions which could take one to a state in which the atom was in its excited state with an even number of photons present, or the atom was in its ground state with an odd number of photons present.

Before studying the general case, it is convenient to consider a simple case, that of spontaneous emission. A study of this situation indicates how to proceed in more complex cases.

4. Spontaneous emission

We assume that the initial state of the system is that in which the atom is in its excited state and there are no photons present. Thus we take

$$|\psi\rangle = |\alpha\rangle|0\rangle, \tag{28}$$

and consequently

$$\left.\begin{aligned} a_n &= \delta_{n,0} \\ b_n &= 0, \qquad \text{all } n \end{aligned}\right\}. \tag{29}$$

Our general equations (18) and (19) now reduce to

$$(E-\omega_0-n\omega)A_n = \delta_{n,0}+g\sqrt{n}B_{n-1}+g^*\sqrt{n+1}B_{n+1}, \qquad n = 0, 2, 4, 6, \ldots \tag{30}$$

and

$$(E-m\omega)B_m = g\sqrt{m}A_{m-1}+g^*\sqrt{m+1}A_{m+1}, \qquad m = 1, 3, 5, 7, \ldots. \tag{31}$$

We have written A_n for α_n and B_m for β_m to distinguish the solution in the case of spontaneous emission from the solutions in the general case.

If we make the substitutions

$$A_n = \frac{\sqrt{n!}g^n}{\lambda_0\mu_1\lambda_2\mu_3\ldots\mu_{n-1}\lambda_n} \qquad (n \text{ even}) \tag{32}$$

and

$$B_m = \frac{\sqrt{m!}g^m}{\lambda_0\mu_1\lambda_2\mu_3\ldots\lambda_{m-1}\mu_m} \qquad (m \text{ odd}) \tag{33}$$

in (30) and (31) we find that the λ_n and μ_m must satisfy

$$\lambda_n = (E-\omega_0-n\omega)-\frac{|g|^2(n+1)}{\mu_{n+1}} \tag{34}$$

and

$$\mu_m = (E-m\omega)-\frac{|g|^2(m+1)}{\lambda_{m+1}} \tag{35}$$

in order that (32) and (33) be solutions of (30) and (31). We may rewrite these as

$$\lambda_n = (E-\omega_0-n\omega)-\cfrac{|g|^2(n+1)}{E-(n+1)\omega-\cfrac{|g|^2(n+2)}{\lambda_{n+2}}} \tag{36}$$

and

$$\mu_m = (E-m\omega)-\cfrac{|g|^2(m+1)}{E-\omega_0-(m+1)\omega-\cfrac{|g|^2(m+2)}{\mu_{m+2}}}. \tag{37}$$

(36) and (37) lend themselves readily to a continued fraction solution (Wall 1948):

$$\lambda_n = E-\omega_0-n\omega-\cfrac{|g|^2(n+1)}{E-(n+1)\omega-\cfrac{|g|^2(n+2)}{E-\omega_0-(n+2)\omega-\cfrac{|g|^2(n+3)}{E-(n+3)\omega-\ldots}}} \tag{38}$$

and

$$\mu_m = E-m\omega-\cfrac{|g|^2(m+1)}{E-\omega_0-(m+1)\omega-\cfrac{|g|^2(m+2)}{E-(m+2)\omega-\cfrac{|g|^2(m+3)}{E-\omega_0-(m+3)\omega-\ldots}}}. \tag{39}$$

Equations (10), (32) and (33) imply that the zeros of λ_n and μ_m are energy eigenvalues of H.

The simplest probability amplitude is A_0, which, according to (32), is just

$$A_0 = \frac{1}{\lambda_0}. \tag{40}$$

This is related to $P_0(t)$, the probability that the atom will be in its excited state at time t with no photons present if it was in such a state at time $t = 0$. From (7), (12), (40) and (38) we have

$$P_0(t) = \left| -\frac{1}{2\pi i}\oint dE \frac{e^{-iEt}}{E-\omega_0-\dfrac{|g|^2}{E-\omega-\dfrac{2|g|^2}{E-\omega_0-2\omega-\dfrac{3|g|^2}{E-3\omega-\ldots}}}} \right|^2. \tag{41}$$

Expression (41) is in a reasonably convenient form for making numerical calculations as one can truncate the continued fraction at the appropriate point to give the desired accuracy. The first approximant (Wall 1948) to the continued fraction in (41) leads to

$$P_0^{(R)}(t) = \left| -\frac{1}{2\pi i}\oint dE \frac{e^{-iEt}}{E-\omega_0-\dfrac{|g|^2}{E-\omega}} \right|^2 \tag{42}$$

which is just the 'exact' solution in the rotating wave approximation (Swain 1972d, equation (53)).

It is interesting to compare the predictions of (41) and (42) for the weak coupling case. The integrals in these equations may be evaluated using the calculus of residues. To zeroth order in $|g|^2$, the poles of the integrand of (41) occur at

$$E = \omega_0, \qquad E = \omega, \qquad E = \omega_0+2\omega, \qquad E = 3\omega, \ldots.$$

In the rotating wave approximation we would get contributions from only the first two poles, which, as is well known (eg Swain 1972c), lead to a cosinusoidal behaviour for $P_0^{(R)}(t)$. If, for example, we took the third approximant to the continued fraction in (41) we would find firstly that the first two poles would occur at values somewhat different to those obtained in the rotating wave approximation, that is, there would be an effective frequency shift, and secondly that there would be two further poles. It is reasonable to assume that if $|g|^2$ was sufficiently small then the residues at the second two poles would be very much smaller than the residues at the first two poles. Consequently (41) would predict an approximately cosinusoidal behaviour of slightly different frequency from that of $P_0^{(R)}(t)$ with additional higher frequency modulations of small amplitude superimposed upon it. Both (41) and (42) predict that $P_0(t)$ should tend to unity at $t = 0$ (because of our initial conditions) so that we would expect the differences between the two formulae to be apparent only at times sufficiently large for the differences between the two frequencies to produce an appreciable effect and/or the effects of the modulations to become large. Hence we would expect significant differences between the predictions of (41) and (42) for sufficiently large times, even if the coupling was weak. Furthermore $P_0^{(R)}(t)$ is a periodic function, whereas (41) in general describes an almost periodic function.

Physically the existence of just two poles in the rotating wave solutions is a consequence of the property that in the rotating wave approximation only such states as $|\alpha\rangle|n\rangle$ and $|\beta\rangle|n+1\rangle$ are connected. If the rotating wave approximation is not made,

then the state $|\alpha\rangle|n\rangle$ is connected to the states $|\beta\rangle|n\pm1\rangle$ directly, and the states $|\alpha\rangle|n\pm2\rangle, |\alpha\rangle|n\pm4\rangle, \ldots, |\beta\rangle|n\pm3\rangle, |\beta\rangle|n\pm5\rangle, \ldots$ indirectly. It is the existence of these additional transitions which makes the discussion of the full interaction more difficult than that of the rotating wave interaction, and which destroys the periodic behaviour typical of the latter case.

5. Stimulated emission and absorption

Here we consider the evolution of the atom from an initial state in which there are s photons present, and the atom is either in its excited state or in its ground state. In the first case the initial behaviour of the system will be principally that of stimulated and spontaneous emission of radiation; for brevity we will refer to this as the case of stimulated emission. In the second case, the behaviour of the system for the first few instants will be in the main stimulated absorption of radiation. The procedure for obtaining the solutions is similar in both cases, so we shall first treat the case of stimulated emission in detail, and then merely present the results for stimulated absorption.

The initial conditions appropriate to stimulated emission imply that

$$\left.\begin{aligned} a_n &= \delta_{n,s} \\ b_n &= 0, \qquad \text{all } n \end{aligned}\right\} \tag{43}$$

so that our difference equations (18) and (19), take the form

$$(E-\omega_0-n\omega)A_n^s(\mathrm{e}) = \delta_{n,s}+g\sqrt{n}B_{n-1}^s(\mathrm{e})+g^*\sqrt{n+1}B_{n+1}^s(\mathrm{e}) \tag{44}$$

$$(E-n\omega)B_n^s(\mathrm{e}) = g\sqrt{n}A_{n-1}^s(\mathrm{e})+g^*\sqrt{n+1}A_{n+1}^s(\mathrm{e}) \tag{45}$$

where the superscript s and the letter e in parenthesis are used to make explicit the fact that $A_n^s(\mathrm{e})$ and $B_n^s(\mathrm{e})$ are the solutions of (18) and (19) appropriate to the initial conditions (43). Prompted by the form of the solutions in the case of spontaneous emission studied in § 4, we adopt the trial solutions

$$A_n^s(\mathrm{e}) = \begin{cases} \dfrac{g^{n-s}A_s^s(\mathrm{e})}{\mu_{s+1}\lambda_{s+2}\mu_{s+3}\cdots\mu_{n-1}\lambda_n}\left(\dfrac{n!}{s!}\right)^{1/2}, & n > s \qquad (46a) \\ \dfrac{(g^*)^{s-n}A_s^s(\mathrm{e})}{l_n m_{n+1} l_{n+2}\cdots l_{s-2}m_{s-1}}\left(\dfrac{s!}{n!}\right)^{1/2}, & n < s \qquad (46b) \end{cases}$$

and

$$B_n^s(\mathrm{e}) = \begin{cases} \dfrac{g^{n-s}A_s^s(\mathrm{e})}{\mu_{s+1}\lambda_{s+2}\mu_{s+3}\cdots\lambda_{n-1}\mu_n}\left(\dfrac{n!}{s!}\right)^{1/2}, & n > s \qquad (47a) \\ \dfrac{(g^*)^{s-n}A_s^s(\mathrm{e})}{m_n l_{n+1} m_{n+2}\cdots l_{s-2}m_{s-1}}\left(\dfrac{s!}{n!}\right)^{1/2}, & s < n \qquad (47b) \end{cases}$$

where the quantities μ_i, λ_i, l_i, m_i and $A_s^s(\mathrm{e})$ are to be determined.

Consider first the case $n > s$. Substituting from (46a) and (47a) into (44) and (45) we find that (46a) and (47a) are solutions providing that the μ_i and λ_i satisfy the equations

$$E-\omega_0-n\omega = \lambda_n+\frac{|g|^2(n+1)}{\mu_{n+1}}; \qquad n = s+2, s+4, s+6, \ldots \tag{48}$$

and

$$E-n\omega = \mu_n+\frac{|g|^2(n+1)}{\lambda_{n+1}}; \qquad n = s+1, s+3, s+5, \ldots. \tag{49}$$

Apart from the conditions on n, these equations are identical with (34) and (35) (which justifies our notation).

Now take the case $n < s$, and substitute from (46*b*) and (47*b*) into (44) and (45). The conditions that (46*b*) and (47*b*) be solutions are that l_i and m_i satisfy the relations

$$\left.\begin{aligned} l_n &= E-\omega_0-n\omega-\frac{|g|^2 n}{m_{n-1}}; \qquad n = s-2, s-4, s-6, \ldots \\ \text{and} \\ m_n &= E-n\omega-\frac{|g|^2 n}{l_{n-1}}; \qquad n = s-1, s-3, s-5, \ldots \end{aligned}\right\} n \geqslant 0 \tag{50, 51}$$

Casting these equations into a different form we obtain

$$l_n = E-\omega_0-n\omega-\cfrac{|g|^2 n}{E-(n-1)\omega-\cfrac{|g|^2(n-1)}{l_{n-2}}} \tag{52}$$

and

$$m_n = E-n\omega-\cfrac{|g|^2 n}{E-\omega_0-(n-1)\omega-\cfrac{|g|^2(n-1)}{m_{n-2}}}. \tag{53}$$

Now (52) and (53) define *finite* continued fractions, so that l_n and m_n may easily be obtained in closed term for small values of n. For example, we find

$$\begin{aligned} l_0 &= E-\omega_0 & m_0 &= E \\ l_1 &= E-\omega_0-\omega-\frac{|g|^2}{E} & m_1 &= E-\omega-\frac{|g|^2}{E-\omega_0} \\ l_2 &= E-\omega_0-2\omega-\cfrac{2|g|^2}{E-\omega-\cfrac{|g|^2}{E-\omega_0}} & m_2 &= E-2\omega-\cfrac{2|g|^2}{E-\omega_0-\omega-\cfrac{|g|^2}{E}} \\ &\ldots & &\ldots. \end{aligned} \tag{54}$$

To complete the solutions for stimulated emission it remains to find the quantity $A_s^s(\mathrm{e})$. Putting $n = s$ in (44) and using (47*a*) for $B_{s+1}^s(\mathrm{e})$ and (47*b*) for $B_{s-1}^s(\mathrm{e})$ we arrive at the equation

$$(E-\omega_0-s\omega)A_s^s(\mathrm{e}) = 1+\frac{|g|^2 s}{m_{s-1}}A_s^s(\mathrm{e})+\frac{|g|^2(s+1)A_s^s(\mathrm{e})}{\mu_{s+1}}. \tag{55}$$

Consequently

$$A_s^s(\mathrm{e}) = \left(E-\omega_0-s\omega-\frac{|g|^2 s}{m_{s-1}}-\frac{|g|^2(s+1)}{\mu_{s+1}}\right)^{-1}. \tag{56}$$

By utilizing equations (34) and (50), $A_s^s(\mathrm{e})$ may be written alternatively as

$$A_s^s(\mathrm{e}) = \left(\lambda_s-\frac{|g|^2 s}{m_{s-1}}\right)^{-1}. \tag{57}$$

or

$$A_s^s(\mathrm{e}) = \left(l_s - \frac{|g|^2(s+1)}{\mu_{s+1}}\right)^{-1}. \tag{58}$$

If we put $s = 0$ in (56) we find that $A_0^0(\mathrm{e}) = A_0 = 1/\lambda_0$. Equations (48), (49) (46*a*) and (47*a*) then give the same results for spontaneous emission that we obtained in § 4.

Thus we have verified that the solutions appropriate to stimulated emission are given by equations (46), (47), and (56), (57) or (58).

As a specific example, take $s = 2$. We then find

$$\begin{aligned}
&A_0^2(\mathrm{e}) = \frac{(g^*)^2\sqrt{2!}}{l_0 m_1} A_2^2(\mathrm{e}), \qquad A_2^2(\mathrm{e}) = \frac{1}{\lambda_2 - 2|g|^2/m_1} \\
&A_4^2(\mathrm{e}) = \frac{g^2\sqrt{4.3}}{\mu_3\lambda_4} A_2^2(\mathrm{e}), \qquad A_6^2(\mathrm{e}) = \frac{g^4\sqrt{6.5.4.3}}{\mu_3\lambda_4\mu_5\lambda_6} A_2^2(\mathrm{e}), \ldots \\
&B_1^2(\mathrm{e}) = \frac{g^*\sqrt{2}}{m_1} A_2^2(\mathrm{e}), \qquad B_3^2 = \frac{g\sqrt{3}}{\mu_3} A_2^2(\mathrm{e}), \ldots \\
&A_1^2(\mathrm{e}) = A_3^2(\mathrm{e}) = A_5^2(\mathrm{e}) = \ldots = B_0^2(\mathrm{e}) = B_2^2(\mathrm{e}) = B_4^2(\mathrm{e}) = \ldots = 0.
\end{aligned} \tag{59}$$

To conclude this section we study the case of stimulated absorption. Taking $|\psi\rangle = |\beta\rangle|s\rangle$ our initial conditions take the form

$$\left.\begin{aligned} a_n &= 0, \qquad \text{all } n \\ b_n &= \delta_{n,s} \end{aligned}\right\} \tag{60}$$

so that the difference equations (18) and (19) may then be written as

$$(E - \omega_0 - n\omega)A_n^s(\mathrm{a}) = g\sqrt{n}B_{n-1}^s(\mathrm{a}) + g^*\sqrt{n+1}B_{n+1}^s(\mathrm{a}) \tag{61}$$

$$(E - n\omega)B_n^s(\mathrm{a}) = \delta_{s,n} + g\sqrt{n}A_{n-1}^s(\mathrm{a}) + g^*\sqrt{n+1}A_{n+1}^s(\mathrm{a}) \tag{62}$$

where the bracketed letter a now indicates that we are dealing with the case of absorption. There is a symmetry between equations (44) and (45) and equations (61) and (62); if we interchange $A_n^s(\mathrm{e})$ and $B_n^s(\mathrm{a})$, $B_n^s(\mathrm{e})$ and $A_n^s(\mathrm{a})$, $E - \omega_0$ and E, then (62) becomes identical to (44), and (61) to (45). We may use this property (and the property that $\lambda_i \leftrightarrow \mu_i$, $l_i \leftrightarrow m_i$ as E and $(E - \omega_0)$ are interchanged) to write down the solutions in the case of absorption as

$$A_n^s(\mathrm{a}) = \begin{cases} \dfrac{g^{n-s}B_s^s(\mathrm{a})}{\lambda_{s+1}\mu_{s+2}\lambda_{s+3}\cdots\mu_{n-1}\lambda_n}\left(\dfrac{n!}{s!}\right)^{1/2}, & n > s \qquad (63a) \\ \dfrac{(g^*)^{s-n}B_s^s(\mathrm{a})}{l_n m_{n+1} l_{n+2}\cdots m_{s-2}l_{s-1}}\left(\dfrac{s!}{n!}\right)^{1/2}, & n < s \qquad (63b) \end{cases}$$

$$B_n^s(\mathrm{a}) = \begin{cases} \dfrac{g^{n-s}B_s^s(\mathrm{a})}{\lambda_{s+1}\mu_{s+2}\lambda_{s+3}\cdots\lambda_{n-1}\mu_n}\left(\dfrac{n!}{s!}\right)^{1/2}, & n > s \qquad (64a) \\ \left(E - s\omega - \dfrac{|g|^2 s}{l_{s-1}} - \dfrac{|g|^2(s+1)}{\lambda_{s+1}}\right)^{-1} \\ \quad = \left(\mu_s - \dfrac{|g|^2 s}{l_{s-1}}\right)^{-1} = \left(m_s - \dfrac{|g|^2(s+1)}{\lambda_{s+1}}\right)^{-1} \end{cases} n = s \qquad (64b)$$

$$\begin{cases} \dfrac{(g^*)^{s-n}B_s^s(\mathrm{a})}{m_n l_{n+1} m_{n+2}\cdots m_{s-2}l_{s-1}}\left(\dfrac{s!}{n!}\right)^{1/2}, & n < s. \qquad (64c) \end{cases}$$

Again taking $s = 2$ as an example, we find

$$B_0^2(a) = \frac{(g^*)^2\sqrt{2}}{m_0 l_1} B_2^2(a), \qquad B_2^2(a) = \left(\mu_2 - \frac{2|g|^2}{l_1}\right)^{-1}$$

$$B_4^2(a) = \frac{g^2\sqrt{4.3}}{\lambda_3 \mu_4} B_2^2(a), \ldots$$

$$A_1^2(a) = \frac{g^*\sqrt{2}}{l_1} B_2^2(a), \qquad A_1^3(a) = \frac{g\sqrt{3}}{\lambda_3} B_2^2(a), \ldots$$

$$B_1^2(a) = B_3^2(a) = B_5^2(a) = \ldots = A_0^2(a) = A_0^4(a) = A_0^6(a) = \ldots = 0. \tag{65}$$

Once the A_n^s and B_n^s are known the corresponding probability amplitudes may be found from (12).

6. General solution of the difference equations

We are now in a position to find the solution of the general difference equations (18) and (19), where a_n and b_n are arbitrary. It is convenient to consider the calculation of the sets

$$(\alpha_0, \alpha_2, \alpha_4, \ldots; \beta_1, \beta_3, \beta_5, \ldots) \tag{26}$$

and

$$(\alpha_1, \alpha_3, \alpha_5, \ldots; \beta_0, \beta_2, \beta_4, \ldots) \tag{27}$$

separately. Examining the case (26) first of all, it is apparent that the α_n and β_n in this set will be functions of the $a_0, a_2, a_4, \ldots; b_1, b_3, b_5, \ldots$ only. Accordingly we take as our trial solution

$$\alpha_n = \sum_{s=0}^{\infty} (a_{2s} u_n^{2s} + b_{2s+1} u_n^{2s+1}) \tag{66}$$

$$\beta_n = \sum_{s=0}^{\infty} (b_{2s+1} v_n^{2s+1} + a_{2s} v_n^{2s}) \tag{67}$$

where the u_n^i and the v_n^i are to be determined. Substituting from (66) and (67) into (18), and rearranging the terms a little, we find

$$\sum_{s=0}^{\infty} [a_{2s}\{(E-\omega_0-n\omega)u_n^{2s} - \delta_{2s,n} - g\sqrt{n}v_{n-1}^{2s} - g^*\sqrt{n+1}v_{n+1}^{2s}\}$$

$$+ b_{2s+1}\{(E-\omega_0-n\omega)u_n^{2s+1} - g\sqrt{n}v_n^{2s+1} - g^*\sqrt{n+1}v_{n+1}^{2s+1}\}] = 0, \tag{68}$$

whereas if we substitute from (66) and (67) into (19) we obtain

$$\sum_{s=0}^{\infty} [a_{2s}\{(E-n\omega)v_n^{2s} - g\sqrt{n}u_{n-1}^{2s} - g^*\sqrt{n+1}u_{n+1}^{2s}\} + b_{2s+1}\{(E-n\omega)v_n^{2s+1} - \delta_{2s+1,n}$$

$$- g\sqrt{n}u_n^{2s+1} - g^*\sqrt{n+1}u_{n+1}^{2s+1}\}] = 0. \tag{69}$$

As the a_{2s} and b_{2s+1} are arbitrary quantities, for equations (68) and (69) to be satisfied in general we must require

$$(E-\omega_0-n\omega)u_n^{2s} = \delta_{2s,n} + g\sqrt{n}v_{n-1}^{2s} + g^*\sqrt{n+1}v_{n+1}^{2s} \qquad (n \text{ even}) \tag{70}$$

$$(E-n\omega)v_n^{2s} = g\sqrt{n}u_{n-1}^{2s} + g^*\sqrt{n+1}u_{n+1}^{2s} \qquad (n \text{ odd}) \tag{71}$$

and

$$(E-\omega_0-n\omega)u_n^{2s+1} = g\sqrt{n}v_{n-1}^{2s+1}+g^*\sqrt{n+1}v_{n+1}^{2s+1} \qquad (n \text{ even}) \tag{72}$$

$$(E-n\omega)v_n^{2s+1} = \delta_{2s+1,n}+g\sqrt{n}u_{n-1}^{2s+1}+g^*\sqrt{n+1}u_{n+1}^{2s+1} \qquad (n \text{ odd}). \tag{73}$$

Now equations (70) and (71) are identical in form with equations (44) and (45), so that we may write at once

$$u_n^{2s} \equiv A_n^{2s}(\mathrm{e}) \tag{74}$$

$$v_n^{2s} \equiv B_n^{2s}(\mathrm{e}) \tag{75}$$

where the $A_n^{2s}(\mathrm{e})$ and $B_n^{2s}(\mathrm{e})$ are given by equations (46), (47) and (56). Equations (72) and (73) are equivalent to equations (61) and (62), so that we have

$$u_n^{2s+1} \equiv A_n^{2s+1}(\mathrm{a}) \tag{76}$$

$$v_n^{2s+1} = B_n^{2s+1}(\mathrm{a}) \tag{77}$$

where the $A_n^{2s+1}(\mathrm{a})$ and $B_n^{2s+1}(\mathrm{a})$ are given by equations (63) and (64).

Thus our general solution in the first case is

$$\alpha_n = \sum_{s=0}^{\infty}(a_{2s}A_n^{2s}(\mathrm{e})+b_{2s+1}A_n^{2s+1}(\mathrm{a})) \qquad (n \text{ even}) \tag{78}$$

$$\beta_n = \sum_{s=0}^{\infty}(b_{2s+1}B_n^{2s+1}(\mathrm{a})+a_{2s}B_n^{2s}(\mathrm{e})) \qquad (n \text{ odd}). \tag{79}$$

One may proceed in a similar manner to find the general solutions for the second set. The result is

$$\alpha_n = \sum_{s=0}^{\infty}(a_{2s+1}A_n^{2s+1}(\mathrm{e})+b_{2s}A_n^{2s}(\mathrm{a})) \qquad (n \text{ odd}) \tag{80}$$

$$\beta_n = \sum_{s=0}^{\infty}(b_{2s}B_n^{2s}(\mathrm{a})+a_{2s+1}B_n^{2s+1}(\mathrm{e})) \qquad (n \text{ even}). \tag{81}$$

The general structure of the solutions is made apparent by giving a few examples. Thus

$$\alpha_0 = a_0A_0^0+b_1A_0^1+a_2A_0^2+b_3A_0^3+\ldots \tag{82}$$

$$\alpha_1 = b_0A_1^0+a_1A_1^1+b_2A_1^2+a_3A_1^3+\ldots \tag{83}$$

etc.

More explicitly, the first few terms are

$$\alpha_0 = \frac{a_0}{\lambda_0}+\frac{b_1g^*}{l_0(\mu_1-|g|^2/l_0)}+\frac{a_2(g^*)^2\sqrt{2!}}{l_0m_1(\lambda_2-2|g|^2/m_1)}+\frac{b_3(g^*)^3\sqrt{3!}}{l_0m_1l_2(\mu_3-3|g|^2/l_2)}+\ldots \tag{84}$$

$$\alpha_1 = \frac{b_0g}{\lambda_1\mu_0}+\frac{a_1}{\lambda_1-|g|^2/m_0}+\frac{b_2g^*\sqrt{2}}{l_1(\mu_2-2|g|^2/l_1)}+\frac{a_3(g^*)^2\sqrt{3\,.\,2}}{l_1m_2(\lambda_3-3|g|^2/m_2)}+\ldots \tag{85}$$

$$\alpha_2 = \frac{a_0g^2\sqrt{2!}}{\lambda_0\mu_1\lambda_2}+\frac{b_1g\sqrt{2}}{\lambda_2(\mu_1-|g|^2/l_0)}+\frac{a_2}{\lambda_2-2|g|^2/m_1}+\frac{b_3g^*\sqrt{3}}{l_2(\mu_3-3|g|^2/l_2)}+\ldots. \tag{86}$$

The expressions for the β_n may be obtained from the expressions for the α_n by making the substitutions

$$a_n \to b_n, \qquad \lambda_i \to \mu_i, \qquad \mu_i \to \lambda_i, \qquad l_j \to m_j, \qquad m_j \to l_j. \tag{87}$$

(84), (85) and (86) may be compared with the exact solutions in the RWA, (24) and (25), which we note may be written in the form

$$\alpha_n^{(\mathrm{R})} = \frac{a_n}{\lambda_n^{(1)}} + \frac{b_{n+1} g\sqrt{n+1}}{\lambda_n^{(1)} m_{n+1}^{(0)}} \tag{88}$$

$$\beta_n^{(\mathrm{R})} = \frac{b_n}{m_n^{(1)}} + \frac{a_{n-1} g\sqrt{n}}{m_n^{(1)} \lambda_{n-1}^{(0)}} \tag{89}$$

where

$$\lambda_n^{(1)} = E - \omega_0 - n\omega - |g|^2 (n+1)/\{E-(n+1)\omega\} \tag{90}$$

$$m_n^{(1)} = E - n\omega - |g|^2 n/\{E-\omega_0-(n-1)\omega\} \tag{91}$$

$$m_{n+1}^{(0)} = E - (n+1)\omega = \mu_{n+1}^{(0)} \tag{92}$$

and

$$\lambda_{n-1}^{(0)} = E-(n-1)\omega = l_{n-1}^{(0)}. \tag{93}$$

$\lambda_n^{(1)}$ is the first approximant to λ_n, $m_{n+1}^{(0)}$ the zeroth approximant to m_{n+1}, etc.

7. Conclusion

We have used the generalized Laplace transform technique to obtain formally exact continued fraction solutions for the time-dependent probability amplitudes of the system consisting of a single photon interacting with a single atom in the electric dipole approximation. The zeros of the continued fractions also give energy eigenvalues of the system. Our equations are in a useful form for numerical computation, as the continued fractions may be terminated at an appropriate stage, and the probability amplitudes then expressed as integrals of explicit functions. In addition, our results are obtained in a form which makes comparison with the solutions in the rotating wave approximation straightforward; it is easy to identify those terms which are omitted when the rotating wave approximation is made.

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